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Zimní semestr 2025/26

Mathematical analysis II

Image and pre-image of mappings

Let X, Y be non-empty sets, $f : X \to Y$ a mapping, and $A \subset X$, $B \subset Y$.

- The image of the set A under the mapping f is the set
$$f[A] = \{f(x) : x \in A\} \subset Y;$$
- The pre-image of the set B under the mapping f is the set
$$f^{-1}[B] = \{x : f(x) \in B\} \subset X.$$

We say that f is

- *onto* (or surjective), if $f[X] = Y$ (resp. $f[A] = B$);
- *one-to-one* (or injective), if for any $x, y \in X,\ x \neq y$, we have $f(x) \neq f(y)$.

Note that one could be attempted to say that f is injective if $X = f^{-1}[Y]$, however, this holds all the time just by the reason that $f[X]$ is a subset of Y, so the whole image of X under f is already part of Y and if we take even more points from Y, nothing changes (we can't have a "larger" X). Example: $X = \{1, 2\}$, $Y = \{1, 2, 3\}$, $f(1) = 1$, $f(2) = 2$. Then $f^{-1}[\{1, 2\}] = X$, but of course also $f^{-1}[Y] = X$, even if $f^{-1}[\{3\}] = \emptyset$. (That $\{3\}$ "does not harm" follows from the fact that $f^{-1}[A_1 \cup A_2] = f^{-1}[A_1] \cup f^{-1}[A_2]$. Try to prove this!)

That a mapping is *onto* can be also expressed like this: For any $y \in Y$, there is some $x \in X$ such that $f(x) = y$ (we can "hit" every $y \in Y$).

The following theorem holds:

Theorem: For any mapping, we have

1) $f[A] \subset B \Leftrightarrow A \subset f^{-1}[B]$
2) $f[f^{-1}[B]] \subset B$ and equality holds if and only if f is *onto*
3) $f^{-1}[f[A]] \supset A$ and equality holds if and only if f is *one-to-one*

We will just show the equality statements in 2 and 3 (it's a good idea to make a sketch):

2) We have to show that if $b \in B$ and f is *onto*, then $b \in f[f^{-1}[B]]$. Since f is *onto*, there is some $a \in f^{-1}[B]$ such that $f(a) = b$. By definition of the image this is nothing else than $b \in f[f^{-1}[B]]$. (Try to take the example from above and find the place where $\{3\}$ in this proof harms.)

3) We have to show that if $a \in f^{-1}[f[A]]$ and f is *one-to-one*, then $a \in A$. Since $a \in f^{-1}[f[A]]$, there is some $b \in f[A]$ such that $f(a) = b$. If now $a \notin A$, then $f(a) \notin f[A]$, because f is *one-to-one* (otherwise there would be another $c \in A$ such that $f(c) = f(a)$, which is impossible). But that's a contradiction, since at the same time we now have $b \in f[A]$ and $b = f(a) \notin f[A]$, such that finally $a \in A$.