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Mathematical analysis II

Voluntary Homework 11

To be voluntarily handed in whenever via OWL

Transformers

Exercise 1 (Isodiametric inequality in 2D).

Let $g : \mathbb{R} \to (0, \infty)$ be continuous and π-periodic, that is, $g(\theta + \pi) = g(\theta)$ for any $\theta \in \mathbb{R}$. We define

$$\Omega = \{(x, y) = (r\cos(\theta), r\sin(\theta)) \in \mathbb{R}^2 : \theta \in [0, 2\pi],\ 0 \le r \le g(\theta)\},$$

and define the diameter of Ω as

$$\operatorname{diam}(\Omega) = \sup\{|x - y| : x, y \in \Omega\}.$$

a) Show that the area A of Ω is given by

$$A = \frac{1}{2}\int_0^{2\pi} [g(\theta)]^2 d\theta.$$

(*Hint*: write $A = \int_\Omega 1\, d(x, y)$, use a coordinate transformation to polar coordinates $(x, y) = (r\cos\phi, r\sin\phi)$, and use Fubini's theorem.)

b) Show that

$$A \le \pi \left(\frac{\operatorname{diam}(\Omega)}{2}\right)^2.$$

Give a geometrical interpretation of this inequality *for fixed diameter* $\operatorname{diam}(\Omega)$. To this end, compare an arbitrary function g with the special constant one $g_0 = \operatorname{diam}(\Omega)/2$. (*Hint*: what geometric property is included in the periodicity condition on g? Which relationship between $\operatorname{diam}(\Omega)$ and $\sup_{\theta\in\mathbb{R}} g(\theta)$ you can conclude from that? A sketch might be helpful.)

Solution. a) We use spherical coordinates $\Phi(r, \theta) = (r\cos\theta, r\sin\theta)$ and calculate the Jacobi determinant as

$$\det D\Phi = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

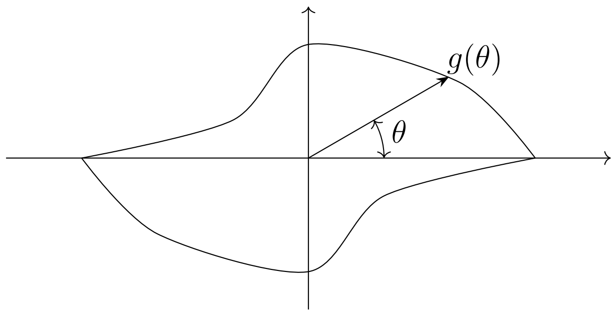
Thus, transformation theorem and Fubini yield

$$A = \int_\Omega 1\, d(x, y) = \int_0^{2\pi}\int_0^{g(\theta)} r\, dr\, d\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_{r=0}^{r=g(\theta)} d\theta = \frac{1}{2}\int_0^{2\pi}[g(\theta)]^2\, d\theta.$$

b) The periodicity of g result in the symmetry of Ω with respect to the origin; in fact, Ω is so-called *star-shaped* with respect to the origin. Since g is continuous and everywhere positive, this yields that $2\sup_{\theta\in\mathbb{R}} g(\theta) = \operatorname{diam}(\Omega)$. Plugging this into A gives

$$A = \frac{1}{2}\int_0^{2\pi} [g(\theta)]^2\, d\theta \le \frac{1}{2}\int_0^{2\pi} \left[\sup_{\theta\in\mathbb{R}} g(\theta)\right]^2 d\theta = \frac{1}{2}\left(\frac{\operatorname{diam}(\Omega)}{2}\right)^2 \cdot 2\pi = \pi\left(\frac{\operatorname{diam}(\Omega)}{2}\right)^2.$$

The interpretation of this inequality is as follows: for fixed diameter, the circle (resp. disc) maximizes the area of a set having this diameter. In other words, any set having s specified diameter cannot have more area than the corresponding disc with the same diameter.



Exercise 2 (Christmas).

We define

$$h(z) = z - \frac{1}{2}\lfloor z\rfloor - \lfloor z/4\rfloor(z - 5/2), \quad z \in (0,5).$$

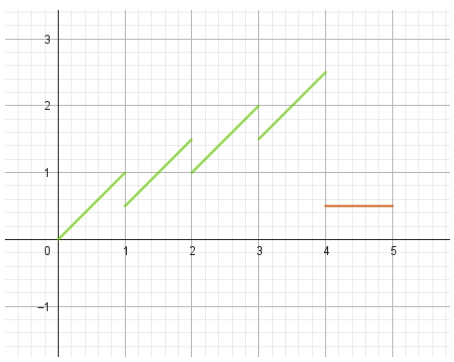
Here, $\lfloor x\rfloor = \max\{k \in \mathbb{Z} : k \le x\}$ is the largest integer less than a real number $x \in \mathbb{R}$. Let further

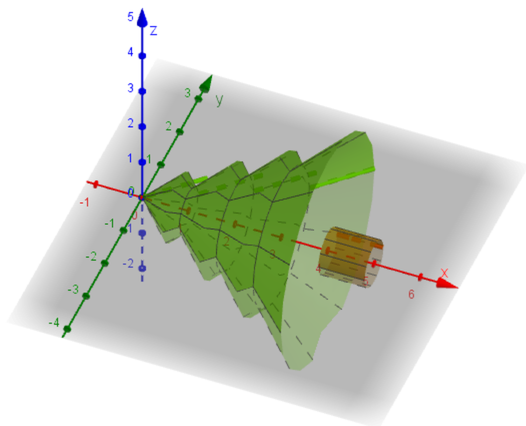
$$CT = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \le h(z)^2\}.$$

a) Sketch h and the set CT (using colors if wished).

b) Calculate the volume of CT.

Solution. a) Pictures:





b) We will use cylindrical coordinates $\Phi(x,y,z) = (r\cos\theta, r\sin\theta, z)$. The Jacobi determinant of this mapping is given by

$$\det D\Phi = \det\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r.$$

Hence, transformation formula yields

$$vol(CT) = \int_{CT} 1\, d(x,y) = \int_0^5 \int_0^{2\pi} \int_0^{h(z)} r\, dr\, d\theta\, dz = \pi \int_0^5 [h(z)]^2\, dz.$$

We split the interval $(0,5)$ into subintervals $(k, k+1)$ with $k \in \{0, ..., 4\}$ and calculate

- $z \in (0,1)$: $h(z) = z$, $\qquad z \in (1,2)$: $h(z) = z - 1/2$,
- $z \in (2,3)$: $h(z) = z - 1$, $\qquad z \in (3,4)$: $h(z) = z - 3/2$,
- $z \in (4,5)$: $h(z) = 1/2$.

This yields

$$\begin{aligned} Vol(WB) &= \pi\left[\int_0^1 z^2 + \int_1^2 (z-1/2)^2 + \int_2^3 (z-1)^2 + \int_3^4 (z-3/2)^2 + \int_4^5 (1/2)^2\right] \\ &= \pi\left[\frac{1}{3}z^3\Big|_0^1 + \frac{1}{3}(z-1/2)^3\Big|_1^2 + \frac{1}{3}(z-1)^3\Big|_2^3 + \frac{1}{3}(z-3/2)^3\Big|_3^4 + \frac{1}{4}\right] \\ &= \pi\left[\frac{1}{3} + \frac{1}{3}\left(\frac{27}{8} - \frac{1}{8}\right) + \frac{1}{3}(8-1) + \frac{1}{3}\left(\frac{125}{8} - \frac{27}{8}\right) + \frac{1}{4}\right] \\ &= \frac{97}{12}\pi. \end{aligned}$$

Merry Christmas and a Happy New Year 2026!