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Mathematical analysis II

Homework 10

To be handed in by Wednesday, 17.12.25, 23:59 h via OWL

Exercise 1 (Fubini's theorem). *(5 points)*

Let $a, b > 0$. Calculate the area of the ellipse

$$E = \left\{x, y \in \mathbb{R} : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\}.$$

Solution. First, we see that $x^2/a^2 \leq 1$ forces $x \in [-a, a]$. Next, we rewrite

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \Leftrightarrow -b\sqrt{1 - \frac{x^2}{a^2}} \leq y \leq b\sqrt{1 - \frac{x^2}{a^2}}.$$

This gives us for any $x \in [-a, a]$ the range of y. Moreover, the set E is compact (it is obviously a closed and bounded subset of $\mathbb{R}^2$, so Heine-Borel theorem applies) and the constant function 1 is continuous, hence, the whole integral $\int_E 1\, d(x, y)$ exists. Thus, Fubini's theorem is applicable and we find for the volume

$$vol(E) = \int_E 1\, d(x, y) \overset{Fubini}{=} \int_{-a}^{a} \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} 1\, dy\, dx = \int_{-a}^{a} 2b\sqrt{1 - \frac{x^2}{a^2}}\, dx.$$

Since $x^2/a^2 \leq 1$, we have $-1 \leq x/a \leq 1$. Thus, we can substitute $x/a = \sin(t)$ with $t \in [-\pi/2, \pi/2]$ such that $dx = a\cos(t)\, dt$ and hence using also $\cos^2(t) = \frac{1}{2}(1 + \cos(2t))$

$$\int_{-a}^{a} 2b\sqrt{1 - \frac{x^2}{a^2}}\, dx = 2b\int_{-\pi/2}^{\pi/2} \sqrt{1 - \sin^2(t)} \cdot a\cos(t)\, dt = 2ab\int_{-\pi/2}^{\pi/2} \cos^2(t)\, dt = \pi ab.$$

In the special case $a = b = r$ this recovers the known formula for the disc: $vol(B_r) = \pi r^2$.

Exercise 2 (A counterexample). *(5 points)*

Let for $x + y \neq 2$

$$f(x, y) = \frac{y - x}{(2 - x - y)^3}.$$

Show (without the help of AI!) that

$$\int_0^1 \int_0^1 f(x, y)\, dx\, dy \neq \int_0^1 \int_0^1 f(x, y)\, dy\, dx.$$

Why this is no contradiction to Fubini's theorem?

Solution. First let us fix y and calculate

$$\int_0^1 \frac{y-x}{(2-x-y)^3}\,dx.$$

There are several ways how to do that; I will use substitution here. To this end, let $z = 2-x-y$, then $dx = -dz$ and the integral turns into

$$\begin{aligned}
\int_0^1 \frac{y-x}{(2-x-y)^3}\,dx &= \int_{1-y}^{2-y} \frac{2y-2+z}{z^3}\,dz = (y-1)\int_{1-y}^{2-y} \frac{2}{z^3}\,dz + \int_{1-y}^{2-y} \frac{1}{z^2}\,dz \\
&= (y-1)\Big(\frac{1}{(1-y)^2} - \frac{1}{(2-y)^2}\Big) + \frac{1}{1-y} - \frac{1}{2-y} \\
&= \frac{y-1+(1-y)}{(1-y)^2} - \frac{y-1+(2-y)}{(2-y)^2} = -\frac{1}{(2-y)^2}.
\end{aligned}$$

For symmetry reasons, for fixed x, a similar calculation leads to

$$\int_0^1 \frac{y-x}{(2-x-y)^3}\,dy = \frac{1}{(2-x)^2}.$$

Hence, we have

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy = -\int_0^1 \int_0^1 f(x,y)\,dy\,dx,$$

and another calculations gives

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy = \int_0^1 -\frac{1}{(2-y)^2}\,dy = \frac{1}{2-y}\bigg|_0^1 = -\frac{1}{2}$$

such that finally

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy = -\frac{1}{2} \neq \frac{1}{2} = \int_0^1 \int_0^1 f(x,y)\,dy\,dx.$$

This outcome does not contradict Fubini's theorem since the integral

$$\int_{[0,1]\times[0,1]} |f(x,y)|\,d(x,y)$$

does not exist (even without the absolute value); even worse, the function f is not bounded on the square $[0,1]\times[0,1]$ since we have

$$f(1,y) = \frac{y-1}{(1-y)^3} = -\frac{1}{(1-y)^2}$$

such that $|f(1,y)| \to \infty$ as $y \to 1$.