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Mathematical analysis II

Homework 9

To be handed in by Wednesday, 10.12.25, 23:59 h via OWL

Exercise 1 (Integration by parts). *(2+3=5 points)*

a) Show that for two continuously differentiable functions $f, g : [a, b] \to \mathbb{R}$, it holds

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx.$$

(*Hint:* consider the function $F(x) = f(x)g(x)$.)

b) Let $f : [a, b] \to \mathbb{R}$ be continuously differentiable and set for any $k \in \mathbb{N}$

$$a_k := \int_a^b f(x)\sin(kx)\,dx.$$

Show that $\lim_{k\to\infty} a_k = 0$.

Solution. a) By product rule, we find

$$F'(x) = f'(x)g(x) + f(x)g'(x).$$

Linearity of the Riemann integral and the fundamental theorem of calculus thus yield

$$\int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx = \int_a^b f'(x)g(x) + f(x)g'(x)\,dx = F(x)\Big|_a^b = f(x)g(x)\Big|_a^b,$$

which implies by re-arranging the desired.

b) This is a version of the so-called Riemann-Lebesgue lemma. Setting $g'(x) = \sin(kx)$ and using part a), we find

$$a_k = -f(x)\frac{\cos(kx)}{k}\Big|_a^b + \int_a^b f'(x)\frac{\cos(kx)}{k}\,dx.$$

Since f is continuously differentiable and $[a, b]$ is compact, there is a constant $M > 0$ such that

$$\sup_{x\in[a,b]} \Big(|f(x) + |f'(x)|\Big) \le M.$$

Using that also $|\cos(kx)| \le 1$, we find

$$|a_k| \le \frac{2M}{k} + \frac{M(b-a)}{k}.$$

Letting $k \to \infty$ yields the desired. Note that we even proved a stronger version: in fact we have a rate of convergence, namely, a_k get not go slower to zero than $1/k$.

Exercise 2 (Riemann integral in nD). *(3+2=5 points)*

The words "interval" and "brick" are used synonymously here.

a) Show *via definition* that for any compact interval (brick) $J \subset \mathbb{R}^n$ and any constant $c \in \mathbb{R}$, the Riemann integral $\int_J c\,dx$ exists and that it holds $\int_J c\,dx = c \cdot vol(J)$.

b) Let $J = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ be a compact interval. A function $\phi : J \to \mathbb{R}$ is called a *step function* if there are constants $c_1, \ldots, c_k \in \mathbb{R}$ and pairwise disjoint intervals (bricks) $I_1, \ldots, I_k \subset J$ such that[1]

$$\phi(x) = \sum_{i=1}^{k} c_i \chi_{I_i}(x), \qquad \text{where} \qquad \chi_{I_i}(x) = \begin{cases} 1 & \text{if } x \in I_i, \\ 0 & \text{else.} \end{cases}$$

Show that for any step function, it holds

$$\int_J \phi(x)\,dx = \sum_{i=1}^{k} c_i \cdot vol(I_i).$$

(You can use without proof that for intervals $I \in \{(a,b),\, [a,b),\, (a,b],\, [a,b]\}$ it holds $vol(I) = b-a$, and that the function χ_{I_i} is integrable on the brick I_i. A sketch might be helpful, say, for $k=2$, $J=[0,1]$, $I_1 = [0, \frac{1}{2}]$, $I_2 = (\frac{1}{2}, 1]$, and $c_1 = 1$, $c_2 = 2$.)

Solution. a) Let P be any partition of the brick J, and let $\mathcal{B}(P)$ be the set of bricks that can be formed from P. Then, we have for the lower and upper sums that

$$s(c,P) = \sum_{B \in \mathcal{B}(P)} \inf_{x \in B}(c) \cdot vol(B) = \sum_{B \in \mathcal{B}(P)} c \cdot vol(B) = c \sum_{B \in \mathcal{B}(P)} vol(B) = c \cdot vol(J),$$
$$S(c,P) = \sum_{B \in \mathcal{B}(P)} \sup_{x \in B}(c) \cdot vol(B) = \sum_{B \in \mathcal{B}(P)} c \cdot vol(B) = c \sum_{B \in \mathcal{B}(P)} vol(B) = c \cdot vol(J),$$

where we used that $vol(J) = \sum_{B \in \mathcal{B}(P)} vol(B)$. Hence, $s(c,P) = S(c,P)$ for any partition, such that this implies that

$$\underline{\int_J} c\,dx = \sup s(c,P) = c \cdot vol(J) = \int S(c,P) = \overline{\int_J} c\,dx.$$

That means that the Riemann integral $\int_J c\,dx$ exists and equals to $c \cdot vol(J)$.

b) For any $i \in \{1, \ldots, k\}$, we have from part a) and the definition of χ_{I_i} that

$$\int_J c_i \chi_{I_i}(x)\,dx = \int_{I_i} c_i\,dx = c_i \cdot vol(I_i).$$

Linearity of the Riemann integral thus forces

$$\int_J \phi(x)\,dx = \sum_{i=1}^{k} \int_J c_i \chi_{I_i}\,dx = \sum_{i=1}^{k} c_i \cdot vol(I_i).$$

Small correction: Indeed $\mathcal{B}(P)$ is not the set of *all* bricks formed from P, but rather *the largest class of (almost) disjoint* bricks formed from P. In other words, bricks in $\mathcal{B}(P)$ are formed just from nearest-neighbour-points from P. For a complete formal definition, see the lecture notes.

[1] Although our definition of the integral uses it, the intervals I_i need not to be closed here.