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Mathematical analysis II

Homework 8

To be handed in by Wednesday, 03.12.25, 23:59 h via OWL

Exercise 1 (Uniform continuity). *(1+3=4 points)*

Let (X,d) and (Y,e) be metric spaces. A function $f : X \to Y$ is said to be *Lipschitz continuous* (resp. just *Lipschitz*) if there exists some $L \geq 0$ such that

$$e(f(x), f(y)) \leq L d(x,y) \quad \forall x, y \in X.$$

a) Show that any Lipschitz continuous function is uniformly continuous.

b) Show that $f : [0,\infty) \to [0,\infty)$, $f(x) = \sqrt{x}$ is uniformly continuous, but not Lipschitz. Here, we use $d(x,y) = e(x,y) = |x-y|$. (*Hint:* Split the nonnegative real line into $[0,2]$ and $[1,\infty)$ and show uniform continuity on each of these intervals. Then argue why this implies uniform continuity everywhere on $[0,\infty)$.)

Exercise 2 (Riemann integral). *(2+2+2=6 points)*

Let $f : [a,b] \to \mathbb{R}$ be integrable (that is, $\int_a^b f(x)\,\mathrm{d}x$ exists and is finite), and set

$$F(x) := \int_a^x f(t)\,\mathrm{d}t.$$

Show:

a) The function f is bounded[1].

b) $F : [a,b] \to \mathbb{R}$ is Lipschitz continuous, i.e., there is some $L \geq 0$ such that

$$|F(x) - F(y)| \leq L|x-y| \quad \forall x, y \in [a,b].$$

c) If $f \geq 0$, then F is monotonically increasing.

[1] This holds generally: any Riemann integrable function is bounded.