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Mathematical analysis II

Homework 7

To be handed in by Wednesday, 26.11.25, 23:59 h via OWL

Exercise 1 (Extrema with constraints). *(6 points)*

Calculate the minimal distance of the point $x_0 = (1, -1, 1)$ to the set $M := \{(x, y, z) \in \mathbb{R}^3 : z^2 = 2xy + 1\}$. (Hint for easier calculations: how to say "náměstí vzdálenosti" in English? Explain why you are allowed to do that/why you don't lose anything.)

Solution. Set

$$\begin{aligned} d(x, y, z) &= |(x, y, z) - (1, -1, 1)| = \sqrt{(x-1)^2 + (y+1)^2 + (z-1)^2}, \\ g(x, y, z) &= z^2 - 2xy - 1. \end{aligned}$$

Obviously $M = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 0\}$. We then want to find the minimum of d under the constraint g. Since d is always nonnegative, we don't lose any information if we square it since squaring is a monotone action ("náměstí vzdálenosti" means "square of distance"); hence, we define $f(x, y, z) = [d(x, y, z)]^2$ and

$$\Lambda(x, y, z; \lambda) = f(x, y, z) + \lambda g(x, y, z) = (x-1)^2 + (y+1)^2 + (z-1)^2 + \lambda(z^2 - 2xy - 1).$$

We then calculate

$$\begin{aligned} \partial_x \Lambda &= 2(x-1) - 2\lambda y, \\ \partial_y \Lambda &= 2(y+1) - 2\lambda x, \\ \partial_z \Lambda &= 2(z-1) + 2\lambda z, \\ \partial_\lambda \Lambda &= z^2 - 2xy - 1 = g(x, y, z). \end{aligned}$$

Finding an extremum needs $\nabla_{(x,y,z,\lambda)} \Lambda = 0$ (note that the gradient is a 4D vector and we will indeed need all four equations). We thus need to solve the system of equations (depending on λ)

$$\begin{aligned} x - 1 - \lambda y &= 0, \\ y + 1 - \lambda x &= 0, \\ z(1+\lambda) - 1 &= 0, \end{aligned} \tag{1}$$

from which we immediately see $z = 1/(1+\lambda)$ and $\lambda \neq -1$. Adding the first and second equation yields $x + y = \lambda(x + y)$ such that $(x + y)(1 - \lambda) = 0$. Thus, either $\lambda = 1$, or $x = -y$. If $\lambda = 1$, then $x = 1 + y$ and $z = \frac{1}{2}$. Inserting this into the constraint g gives

$$0 = g(1 + y, y, \frac{1}{2}) = \frac{1}{4} - 2y(1 + y) - 1 = -\frac{3}{4} - 2y(1 + y).$$

This equation turns into $0 = y^2 + y + \frac{3}{8} = (y + \frac{1}{2})^2 + \frac{1}{8}$, which does not have a real solution. Hence, $\lambda \neq 1$.

If $x = -y$, then the first two equations of (1) yield $x = 1/(1+\lambda)$ and $y = -1/(1+\lambda)$. Inserting into the constraint g yields

$$0 = g(\frac{1}{1+\lambda}, -\frac{1}{1+\lambda}, \frac{1}{1+\lambda}) = \frac{1}{(1+\lambda)^2} + \frac{2}{(1+\lambda)^2} - 1 = \frac{3}{(1+\lambda)^2} - 1$$

such that $\lambda = \pm\sqrt{3} - 1$. Our points of interest are then

$$P_\pm = \pm\frac{1}{\sqrt{3}}(1, -1, 1) \in M.$$

Calculating both squared distances yields

$$f(P_+) = (\frac{1}{\sqrt{3}} - 1)^2 + (-\frac{1}{\sqrt{3}} + 1)^2 + (\frac{1}{\sqrt{3}} - 1)^2 = 3(\frac{1}{\sqrt{3}} - 1)^2 = (\sqrt{3} - 1)^2,$$
$$f(P_-) = (-\frac{1}{\sqrt{3}} - 1)^2 + (\frac{1}{\sqrt{3}} + 1)^2 + (-\frac{1}{\sqrt{3}} - 1)^2 = 3(\frac{1}{\sqrt{3}} + 1)^2 = (\sqrt{3} + 1)^2$$

such that P_+ is the minimum of f and thus of d on M. The minimal distance is then $d(P_+) = \sqrt{3} - 1$.

Exercise 2 (Directional derivative). *(2+2=4 points)*

a) Let $v \in \mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be totally differentiable in the point $x_0 \in \mathbb{R}^n$. Show that $D_v f(x_0) = \nabla f(x_0) \cdot v$.

b) Calculate $D_v f(x_0)$ once using the definition and once with the help of the identity from part a):

$$f(x, y) = x^2 + y^2, \qquad x_0 = (1, 1), \qquad v = (1, 1).$$

Solution. a) Since f is totally differentiable, we can write

$$f(x_0 + tv) - f(x_0) = \nabla f(x_0) \cdot tv + |tv|\mu(tv)$$

for $t \neq 0$ small enough, where μ is a continuous function with $\mu(0) = 0$. From the definition of total differential, this is nothing else than setting $h = tv$. Hence,

$$D_v f(x_0) = \lim_{t\to 0} \frac{f(x_0 + tv) - f(x_0)}{t} = \lim_{t\to 0} \frac{\nabla f(x_0) \cdot tv + |tv|\mu(tv)}{t} = \nabla f(x_0) \cdot v + |v| \lim_{t\to 0} \frac{|t|}{t}\mu(tv).$$

Since $||t|/t| = 1$ and μ is continuous with $\mu(0) = 0$, the last limit is zero and we are done.

b) Via definition: we have

$$f(x_0 + tv) - f(x_0) = (1 + t)^2 + (1 + t)^2 - 2 = 4t + 2t^2;$$

thus

$$D_v f(x_0) = \lim_{t\to 0} \frac{f(x_0 + tv) - f(x_0)}{t} = \lim_{t\to 0} \frac{4t + 2t^2}{t} = 4.$$

Via identity: we have

$$\nabla f(x, y) = (2x, 2y) \Rightarrow \nabla f(x_0) = (2, 2).$$

Thus

$$D_v f(x_0) = \nabla f(x_0) \cdot v = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 + 2 = 4.$$