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Mathematical analysis II

Homework 6

To be handed in by Wednesday, 19.11.25, 23:59 h via OWL

Exercise 1 (Implicitness and more). *(3+1*+3+1=7+1* points)*

Justify all your answers in this task well (i.e., verify assumptions of Theorems you use, conclusions from them, etc.)!

a) Show that close to $x = 0$, there exist a $\delta > 0$ and a continuously differentiable function $g : (-\delta, \delta) \to \mathbb{R}$ such that

$$g(x) = [g(x)]^3 + 2e^{g(x)} \sin(x).$$

**Bonus:* How many such functions exist (for $\delta > 0$ but suitably small)?

b) For an interval $I \subseteq (-\delta, \delta)$, the graph of a function $f : I \to \mathbb{R}$ is defined as

$$\mathrm{gr}(f) = \{(x, f(x)) : x \in I\}.$$

Show that for any compact interval $I \subset (-\delta, \delta)$ and any continuous function $f : I \to \mathbb{R}$, the graph $\mathrm{gr}(f)$ is compact in $\mathbb{R}^2$. What can you say about the graphs $\mathrm{gr}(g)$ and $\mathrm{gr}(g')$ on such intervals I? (Hint: consider the function $\Phi : I \to \mathbb{R}^2$, $\Phi(x) = (x, f(x))$. Which properties Φ has?)

c) What can you say about the monotonicity of g in $x = 0$?

Solution. a) Define the function

$$F(x, y) = y^3 + 2e^y \sin(x) - y.$$

Obviously F is continuous. Then we seek for $\delta > 0$ and a function $g : (-\delta, \delta) \to \mathbb{R}$ such that $F(x, g(x)) = 0$ for any $|x| < \delta$. First, we see $F(0, 0) = 0$ such that our point of interest is $(x_0, y_0) = (0, 0)$. Next, we have

$$\partial_x F = 2e^y \cos(x), \qquad \partial_y F = 3y^2 + 2e^y \sin(x) - 1,$$

such that both partial derivatives are continuous (this yields the continuously differentiability of g), and moreover $\partial_y F(0, 0) = -1 \neq 0$, so IFT is applicable and gives precisely the existence of $\delta > 0$ and g fulfilling the desired.

**Bonus:* If $x = 0$, then

$$F(0, y) = y^3 - y = y(y^2 - 1).$$

In turn, the points $y_0 \in \mathbb{R}$ where $F(0, y_0) = 0$ are precisely $y_0 \in \{-1, 0, 1\}$. Moreover $\partial_y F(0, \pm 1) = 2 \neq 0$ such that IFT is applicable. Thus, there are exactly three such functions g, depending on which value for y_0 we chose at the beginning.

b) Since f is continuous, the function Φ is as well. Thus, the image $\Phi[I]$ is compact since I is (images of compact sets under continuous functions are compact). By IFT, also g is continuous, and since F is continuously differentiable, the continuity transfers also to g'. Hence, both $\mathrm{gr}(g)$ and $\mathrm{gr}(g')$ are compact on compact intervals.

c) Again by IFT, we have

$$g'(x) = -\frac{\partial_1 F(x, g(x))}{\partial_2 F(x, g(x))} = -\frac{2e^{g(x)} \cos(x)}{3[g(x)]^2 + 2e^{g(x)} \sin(x) - 1}.$$

Thus, by $g(0) = 0$, we have $g'(0) = 2 > 0$, and hence g is (strictly) increasing in $x = 0$. (If you chose $y_0 = \pm 1$, then $g'(0) = -e^{\pm 1}$ and hence these functions are strictly *decreasing* in $x = 0$.)

Exercise 2 (Taylor in higher dimensions). *(2+1=3 points)*

Sometimes, Taylor polynomials in higher dimensions can be easier obtained than just using the formula

$$T^n_{f;x_0}(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) + \frac{1}{2}(x - x_0) \cdot [H_f(x_0)(x - x_0)] + ...$$

a) Calculate the Taylor polynomials of $\log(1 + x)$ and $\cos(x)$ around $x = 0$ up to second order.

b) Conclude that for the function

$$f(x, y, z) = (x + 1) \log(y + 1) \cos(z),$$

the Taylor polynomial of second order around $x_0 = (0, 0, 0)$ is given by

$$T^2_{f;x_0}(x) = y + xy - \frac{1}{2}y^2.$$

Solution. a) First, we have $\log(1) = 0$ and $\cos(0) = 1$. Next, $[\log(1+x)]' = \frac{1}{1+x}$, $[\log(1+x)]'' = -\frac{1}{(1+x)^2}$, and $[\cos(x)]' = \sin(x)$, $[\cos(x)]'' = -\cos(x)$, hence

$$T^2_{\log(1+x);0}(x) = x - \frac{x^2}{2}, \qquad T^2_{\cos(x);0}(x) = 1 - \frac{x^2}{2}.$$

b) A product of Taylor polynomials is again a Taylor polynomial, if we talk about the same degree. Thus, we find close to $(0, 0, 0)$

$$(x + 1) \log(1 + y) \cos(z) \approx (x + 1)(y - \frac{y^2}{2})(1 - \frac{z^2}{2}) = xy + y - \frac{xy^2}{2} - \frac{y^2}{2} - \frac{1}{2}z^2(x + 1)(y - \frac{y^2}{2}).$$

(Note that replacing the first $\approx$ with $=$ is *not* correct (why?).) The last member of the product has degree at least 3 and hence it does not belong to $T^2_{f;x_0}$. The same holds for the third member; hence, we indeed find

$$T^2_{f;x_0}(x) = y + xy - \frac{1}{2}y^2.$$