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Mathematical analysis II

Homework 5

To be handed in by Wednesday, 12.11.25, 23:59 h via OWL

Explicitly implicit

Exercise 1. *(3+2=5 points)*

Let the function $F : \mathbb{R} \times (0, \infty) \to \mathbb{R}$, $F(x, y) = ye^x - x\ln(y) - 1$ and the point $P = (0, 1)$ be given.

a) Show that there exists a neighborhood U of P and a function $g : U \to \mathbb{R}$ such that we can write $y = g(x)$ in U. Show also that the function g is continuously differentiable.

b) Calculate $g'(x)$ and $g'(0)$.

Solution. (There was a small mistake in the domain of definition of F, which I corrected in here.)

a) First, we calculate

$$\partial_y F = e^x - \frac{x}{y}$$

such that we see that $\partial_y F(0, 1) = 1 \neq 0$. Moreover, $\partial_x F = ye^x - \ln(y)$ such that both partial derivatives are continuous. Therefore, by the implicit function theorem (IFT), there is some neighborhood U of the point P and a function $g : U \to \mathbb{R}$ with $g(0) = 1$ such that $F(x, g(x)) = 0$ for any $x \in U$ and, again by IFT, g is continuously differentiable in U.

b) By applying chain rule to $F(x, g(x)) = 0$, we obtain

$$\partial_1 F(x, g(x)) + \partial_2 F(x, g(x)) \cdot g'(x) = 0$$

(here I replaced ∂_x and ∂_y by ∂_1 and ∂_2, respectively, to emphasize that I take derivatives wrt. first and second variable). Resolving leads to

$$g'(x) = -\frac{\partial_1 F(x, g(x))}{\partial_2 F(x, g(x))}$$

and the denominator is nonzero in U. This brings us to

$$g'(x) = -\frac{g(x)e^x - \ln(g(x))}{e^x - \frac{x}{g(x)}}$$

and hence

$$g'(0) = -\frac{g(0)e^0 - \ln(g(0))}{e^0 - \frac{0}{g(0)}} = -1.$$

Exercise 2. *(2+2+1=5 points)*

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be given by

$$F(x, y, z) = x^3 - y^3 + z^3 + 2z^2 - 3xyz.$$

a) Show that there is a neighborhood of the point $P = (x_0, y_0) = (1, -1)$ and a function $g = \mathbb{R}^2 \to \mathbb{R}$ with $g(1, -1) = -1$ such that $F(x, y, g(x, y)) = 0$ in this neighborhood.

b) Show that g has a stationary point in $(1, -1)$. (Hint: partial derivatives wrt. x and y and chain rule.)

c) Calculate the tangent plane of g in the point $(1, -1)$. Give a *geometric* explanation why your result is not surprising ("gradient is zero" does not count as geometric).

Solution. a) We calculate similarly to before $F(1, -1, -1) = 1 + 1 - 1 + 2 - 3 = 0$ and

$$\begin{aligned}
\partial_x F &= 3x^2 - 3yz,\\
\partial_y F &= -3y^2 - 3xz,\\
\partial_z F &= 3z^2 + 4z - 3xy
\end{aligned}$$

such that all partial derivatives are continuous. Moreover $\partial_z F(1, -1, -1) = 2 \neq 0$ and hence IFT tells us that there is some neighborhood U of P and a continuously differentiable function $g : U \to \mathbb{R}$ such that $F(x, y, g(x, y)) = 0$.

b) Again using chain rule gives

$$\begin{aligned}
\partial_x F(x, y, g(x, y)) &= \partial_1 F(x, y, g(x, y,)) + \partial_3 F(x, y, g(x, y)) \cdot \partial_x g(x, y) = 0,\\
\partial_y F(x, y, g(x, y)) &= \partial_2 F(x, y, g(x, y,)) + \partial_3 F(x, y, g(x, y)) \cdot \partial_y g(x, y) = 0
\end{aligned}$$

such that the gradient $\nabla g = (\partial_x g, \partial_y g)$ can be written as

$$\begin{aligned}
\nabla g(x, y) &= -[\partial_3 F(x, y, g(x, y))]^{-1} \nabla_{(1,2)} F(x, y, g(x, y))\\
&= -[3g(x, y)^2 + 4g(x, y) - 3xy]^{-1}(3x^2 - 3yg(x, y),\ -3y^2 - 3xg(x, y)),
\end{aligned}$$

where $\nabla_{(1,2)}$ denotes the derivatives wrt. first and second variable of F. In another (maybe clearer) form, this is

$$\partial_x g(x, y) = -\frac{\partial_1 F(x, y, g(x, y))}{\partial_3 F(x, y, g(x, y))}, \qquad \partial_y g(x, y) = -\frac{\partial_2 F(x, y, g(x, y))}{\partial_3 F(x, y, g(x, y))}.$$

To have a stationary point, we need $\nabla g = 0$; thus

$$\begin{aligned}
\nabla g(1, -1) &= -[3g(1, -1)^2 + 4g(1, -1) + 3]^{-1}(3 + 3g(1, -1),\ -3 - 3g(1, -1))\\
&= -\frac{1}{2}(0, 0) = (0, 0),
\end{aligned}$$

hence g has indeed in $(1, -1)$ a stationary point.

c) The tangent plane is given by

$$\tau_{g;P}(x, y) = g(P) + \nabla g(P) \cdot \big((x, y) - P\big).$$

Since $\nabla g(P) = (0, 0)$, we are left with

$$\tau_{g;P}(x, y) = g(P) = -1.$$

In particular, the tangent plane is parallel to the $x - y-$plane. This is not surprising since g has in P a stationary point; hence, the tangent plane must not have a slope there, which means precisely to be parallel to the $x - y-$plane.