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Mathematical analysis II

Homework 5

To be handed in by Wednesday, 05.11.25, 23:59 h via OWL

Exercise 1 (Compactness). *(2+2=4 points)*

a) Let X be a non-empty set and

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{else.} \end{cases}$$

Show that a set $A \subset X$ is compact if and only if it is finite. (Hint: you might prove one direction directly, the other one by contradiction.)

b) Show that the union of finitely many compact sets is again compact. Show via an example that for general (infinite) unions, this fails to be true.

Solution. The empty set is obviously compact, hence, from now on, we assume $A \neq \emptyset$.

a) $\Leftarrow$ Let A be finite and $(x_n)_n$ be a sequence in A. Since A is finite, there must be at least one point $a \in A$ that gets hit by infinitely many x_n. Choose this (now constant) subsequence to find a subsequence that is (obviously) convergent; hence A is compact. (Remark: note that in this direction, we never used the metric d explicitly, and hence we can state: any finite subset of any metric space is compact.)
$\Rightarrow$ Assume A is compact but not finite. Then we can built a sequence $(x_n)_n$ in A such that $x_n \neq x_m$ for any $n \neq m$. Since A is compact, there is a convergent subsequence $(x_{n_k})_k$; but now any convergent sequence in the metric d is constant from some point on, that is, there is some $K \geq 1$ such that for each $k \geq K$, we have $x_{n_k} = x_K$. This is a contradiction to the choice of the sequence; hence A must be finite.

Remark: Note in particular that in this metric, *any* subset is closed and bounded, but just *finite* sets are compact. Compare this with the theorem connecting closedness, boundedness, and compactness from the lecture; what is the crucial point there?

b) Let $(U_i)_{i=1}^k$ be finitely many compact sets and $A = \bigcup_{i=1}^k U_i$ be their union, and let $(x_n)_n$ be a sequence in A. Since the union (better to say, k) is finite, there must exist some set U_j such that infinitely many members of the sequence $(x_n)_n$ lie inside U_j. But now this U_j is compact, so these infinitely many members have a convergent subsequence, which at the same time is also a subsequence of the whole sequence $(x_n)_n$ we started with. Hence, the sequence $(x_n)_n$ has a convergent subsequence in $U_j \subset A$; thus A is compact. For general unions, this fails, take e.g. $U_i = [-i, i]$ for $i \in \mathbb{N}$, then of course each U_i is compact, but $\bigcup_{i\in\mathbb{N}} U_i = \mathbb{R}$ is not.

Another example would be an infinite set X with the metric d from exercise 1. Then every one-point set $\{x\}$ is compact since it is finite, but if we take infinitely many $x_i \in X$, then their union is infinite and thus not compact.

Exercise 2 (Completeness). *(1+2+3=6 points)*

a) Give an example of a metric space (X, d) different from part c) that is not complete.

b) Prove or disprove: There is a non-empty set M such that for any metric d, the metric space (M, d) is not complete.

c) We already know from the lecture that the metric space $(\mathbb{R}, |x - y|)$ is a complete metric space. The aim of this task is to emphasize that the property of being complete depends on the chosen metric:
Let

$$d(x, y) = |\arctan(x) - \arctan(y)|.$$

You can use without proof that this function defines a metric on $\mathbb{R}$. Show that the space $(\mathbb{R}, d)$ is not complete. Here $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is the inverse function of tan. (Hint: investigate the sequence $x_n = n$ for each $n \in \mathbb{N}$.)

Solution. a) Take, e.g., $(X, d) = (\mathbb{Q}, |x - y|)$. As we know, there is a sequence of rationals $(q_n)_n$ that converges to $\sqrt{2} \notin \mathbb{Q}$.

b) This statement is false. Let $M \neq \emptyset$ and take the metric from Exercise 1, then the space (M, d) is complete. Indeed, if $(x_n)_n$ is a Cauchy sequence in (M, d), then from some point on it is constant (see solution to part a) of Exercise 1). Thus, the limit is already a member of the sequence and hence inside M; in other words, any Cauchy sequence converges in M, thus M is complete (independently of how it exactly looks like).

c) The sequence $x_n = n$ for any $n \in \mathbb{N}$ is a Cauchy sequence in $(\mathbb{R}, d)$ that does not converge. To see this, let $m, n \in \mathbb{N}$ with $m > n$. Keeping n fixed and using continuity of $|\cdot|$ and arctan, we have

$$\lim_{m\to\infty} d(n, m) = |\arctan(n) - \frac{\pi}{2}|.$$

Taking now the limit wrt. n we see that

$$\lim_{n\to\infty} |\arctan(n) - \frac{\pi}{2}| = 0,$$

see also Exercise 1 b) from HW2 for the metric $|x - y|$. This in particular means:
Let $\epsilon > 0$ be arbitrary, then there is some $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$|\arctan(n) - \frac{\pi}{2}| < \varepsilon.$$

For any $m > n \geq N$ this then yields

$$d(n, m) \leq |\arctan(n) - \frac{\pi}{2}| + |\frac{\pi}{2} - \arctan(m)| < \varepsilon + \varepsilon = 2\varepsilon$$

and hence the sequence is Cauchy. It is however not convergent in $(\mathbb{R}, d)$ since the only possible limit point would be $+\infty$, however, of course, $\infty \notin \mathbb{R}$; thus the metric space is not complete.