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Mathematical analysis II

Homework 3

To be handed in by Wednesday, 29.10.25, 23:59 h via OWL

Exercise 1 (Chain rule and extrema). *(2+2+2=6 points)*

a) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Show that for $z = f(\frac{xy}{x^2+y^2})$ it holds

$$x\partial_x z + y\partial_y z = 0.$$

b) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ be constant. Calculate the gradient of the function $g : \mathbb{R}^n \to \mathbb{R}$, $g(x) = f(\langle a, x\rangle + b)$. Here, the notation $\langle a, x\rangle$ is the scalar product of these vectors. Remember that the gradient is defined as $\nabla g = (\partial_{x_1} g, ..., \partial_{x_n} g)$.

c) Let

$$f(x, y) = (x - y - 1)^2.$$

Find and classify all extrema (minimum/maximum/saddle point).

Solution. a) We have

$$\partial_x z = f'\left(\frac{xy}{x^2+y^2}\right)\frac{y(x^2+y^2) - 2x^2y}{(x^2+y^2)^2},$$
$$\partial_y z = f'\left(\frac{xy}{x^2+y^2}\right)\frac{x(x^2+y^2) - 2y^2x}{(x^2+y^2)^2}.$$

In turn,

$$\begin{aligned} x\partial_x z + y\partial_y z &= f'\left(\frac{xy}{x^2+y^2}\right)\left[x\frac{y(x^2+y^2) - 2x^2y}{(x^2+y^2)^2} + y\frac{x(x^2+y^2) - 2y^2x}{(x^2+y^2)^2}\right] \\ &= f'\left(\frac{xy}{x^2+y^2}\right)\frac{x^3y + xy^3 - 2x^3y + x^3y + xy^3 - 2xy^3}{(x^2+y^2)^2} = 0. \end{aligned}$$

There is another nice proof that avoids fractions: define $u(x, y) = \frac{xy}{x^2+y^2}$. Then for any $k \neq 0$ we have $u(kx, ky) = u(x, y)$. Now fix (x, y) and define for these (now fixed) points the function $g(k) = u(kx, ky)$. But then $g(k) = u(kx, ky) = u(x, y)$, so g is a constant function of k (since (x, y) is fixed). This means $g'(k) = 0$, where the derivative is wrt. k. Now lets apply chain rule:

$$0 = g'(k) = x\partial_x u(kx, ky) + y\partial_y u(kx, ky).$$

Taking the above equality in $k = 1$ and noting also that, again by chain rule, we have $x\partial_x z + y\partial_y z = f'(u)(x\partial_x u + y\partial_y u)$, this shows the desired.

b) According to chain rule, we get

$$\nabla g(x) = f'(\langle a, x\rangle + b)\nabla(\langle a, x\rangle + b) = f'(\langle a, x\rangle + b)\nabla\langle a, x\rangle$$

since b is constant and hence $\nabla b = 0$. Further for any $i \in \{1, ..., n\}$

$$\partial_{x_i}\langle a, x\rangle = \partial_{x_i}\sum_{j=1}^{n} a_j x_j = a_i$$

such that simply $\nabla\langle a, x\rangle = a$ and thus

$$\nabla g(x) = f'(\langle a, x\rangle + b)a.$$

Note that since a is a vector, also the right hand-side is. Compare this with the one-dimensional rule $[f(ax + b)]' = af'(ax + b)$. (Note: to be fully correct we would need to write $\nabla g(x) = f'(\langle a, x\rangle + b)a^T$ since ∇g is a row vector; but as I said in class, there is not really a common style how to write it.)

c) We calculate

$$\nabla f(x, y) = (2(x - y - 1), -2(x - y - 1))$$

and thus $\nabla f = 0$ iff $y = x - 1$. Since f is quadratic, in particular $f(x, y) \geq 0$ for any pair $(x, y) \in \mathbb{R}^2$, and $f(x, x - 1) = 0$, all points of the form $(x, x - 1)$ are (global) minima. (And, since points $(x, x - 1)$ are the only critical points who are all minima, no maxima or saddle points exist.)

Exercise 2 (Derivatives of higher order). *(2+2=4 points)*

a) Calculate for the following function the derivatives up to second order:

$$f(x, y) = (x + y + 3)^2 - e^{2x+y^2}.$$

b) For the vectorial function $v = (v_1, v_2, v_3) : \mathbb{R}^3 \to \mathbb{R}^3$ we define the divergence and curl via

$$\operatorname{div} v = \nabla \cdot v = \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3,$$
$$\operatorname{curl} v = \nabla \times v = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix}.$$

(Here $\partial_i v_j$ means $\partial v_j/\partial x_i$ for any $i, j \in \{1, 2, 3\}$.) Show that $\operatorname{div}\operatorname{curl} v = 0$.

Solution. a) We have

$$\partial_x f = 2(x + y + 3) - 2e^{2x+y^2}, \qquad \partial_y f = 2(x + y + 3) - 2ye^{2x+y^2},$$
$$\partial^2_{xy} f = 2 - 4ye^{2x+y^2} = \partial^2_{yx} f,$$
$$\partial^2_{xx} f = 2 - 4e^{2x+y^2}, \qquad \partial^2_{yy} f = 2 - 2e^{2x+y^2} - 4y^2 e^{2x+y^2}.$$

Here, we also have $\partial^2_{xy} f = \partial^2_{yx} f$ by Schwarz' theorem (note that the second mixed partial derivatives are continuous).

b) Calculation and re-arranging leads to

$$\begin{aligned}\operatorname{div}\operatorname{curl} v &= \partial_1(\partial_2 v_3 - \partial_3 v_2) + \partial_2(\partial_3 v_1 - \partial_1 v_3) + \partial_3(\partial_1 v_2 - \partial_2 v_1)\\ &= \partial_{12}^2 v_3 - \partial_{21}^2 v_3 + \partial_{23}^2 v_1 - \partial_{32}^2 v_1 + \partial_{31}^2 v_2 - \partial_{13}^2 v^2 = 0.\end{aligned}$$

Especially here, we used Schwarz' theorem to see that $\partial_{ij}^2 v_k = \partial_{ji}^2 v_k$. In a similar way one can show that $\operatorname{curl}\nabla f = 0$ for any (differentiable) $f : \mathbb{R}^3 \to \mathbb{R}$. Such things are very often used in physics, mainly when talking about magnetic fields, and in some special cases, also the other direction holds: if $v : U \to \mathbb{R}^3$ is a function, where $U \subset \mathbb{R}^3$ is “not too bad”, and if $\operatorname{div} v = 0$, then there exists some vector-valued function $w : U \to \mathbb{R}^3$ such that $v = \operatorname{curl} w$. For instance the magnetic field B is always source-free (there are no magnetic monopoles), which means exactly $\operatorname{div} B = 0$, and there exists some so-called vector potential A such that $B = \operatorname{curl} A$. From this the whole theory about magnetic fields follows, how stars are working, etc.