Dr. Florian Oschmann

Karlova univerzita
Zimní semestr 2025/26

Mathematical analysis II

Homework 2

To be handed in by Wednesday, 22.10.25, 23:59 h via OWL

Exercise 1 (Continuity and closedness). *(2+2+2=6 points)*

a) We define the following function $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ via

$$f(x,y) = \frac{x^2 y}{x^2 + y^2}.$$

Decide whether this function is continuous, and whether or not we can extend it to a function that is continuous on the whole of $\mathbb{R}^2$. *(Hint: You can use without proof that for any $x, y \neq 0$, it holds $|xy|/(x^2+y^2) \leq 1$.)*

b) Let (X,d) be a metric space and $a \in X$. Show that the function

$$f : X \to [0,\infty), \qquad f(x) = d(x,a)$$

is continuous.

c) Let (X,d) be a metric space and $f : X \to \mathbb{R}$ be continuous. Show that the kernel

$$\ker f := f^{-1}(\{0\})$$

is closed in X. *(Reminder: for a set $A \subset \mathbb{R}$, the pre-image is defined as $f^{-1}(A) = \{x \in X : f(x) \in A\}$.)*

Solution. a) Outside $(0,0)$ the function is continuous because it's a composition of continuous functions. In the interesting point $(0,0)$ we have, thanks to the inequality given in the hint, that

$$\lim_{(x,y)\to(0,0)} |f(x,y)| \leq \lim_{(x,y)\to(0,0)} |x| = 0.$$

Hence we may define

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{pro } (x,y) \neq (0,0), \\ 0 & \text{pro } (x,y) = (0,0), \end{cases}$$

thus, we can extend f to a function continuous on the whole of $\mathbb{R}^2$.

b) Since d is a metric, triangle inequality holds, in particular,

$$|d(x,a) - d(y,a)| \leq d(x,y)$$

for any $x, y \in X$, which follows from the usual triangle inequality

$$d(x,a) \leq d(x,y) + d(y,a)$$

and symmetry in (x, y). Let now be $x \in X$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence that converges to x, i.e., $d(x, x_n) \to 0$ when $n \to \infty$. But this means that

$$|d(x, a) - d(x_n, a)| \le d(x, x_n) \to 0$$

and hence $d(x_n, a) \to d(x, a)$, which is nothing else than continuity of f. (In this example it holds even more, namely that f is so-called Lipschitz continuous.) The $\varepsilon - \delta$ criterion might here be used as well, try to write it down.

c) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\ker f$ that converges to $x \in X$, in particular, we have $f(x_n) = 0$ for any $n \in \mathbb{N}$. We need to show that also $x \in \ker f$, i.e., $f(x) = 0$. Since f is continuous, we find

$$0 = \lim_{n\to\infty} f(x_n) \overset{continuity}{=} f(\lim_{n\to\infty} x_n) = f(x),$$

and can end the proof. *Note:* Since now we know this property, here we needed (and implicitly assumed) that X is complete, otherwise (x_n) might not have a limit in X.

An even easier proof uses topology: Since f is continuous and $\{0\} \subset \mathbb{R}$ is closed, the pre-image $f^{-1}(\{0\})$ is closed as well. This proof works even without the assumption of completeness.

Exercise 2 (Total vs. partial derivatives). *(4 points)*

Show that the function defined via

$$f(x, y) = \begin{cases} (x^2 + y^2) \sin \frac{1}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{else} \end{cases}$$

is totally differentiable, but its partial derivatives are not continuous.

Solution. With the help of the lecture's theorem we know that if a function has partial derivatives that are continuous, it is also totally differentiable. Outside $(0, 0)$ we calculate

$$\partial_x f = 2x \sin \frac{1}{x^2 + y^2} - \frac{2x}{x^2 + y^2} \cos \frac{1}{x^2 + y^2},$$
$$\partial_y f = 2y \sin \frac{1}{x^2 + y^2} - \frac{2y}{x^2 + y^2} \cos \frac{1}{x^2 + y^2},$$

and these derivatives are obviously continuous outside of $(0, 0)$, hence f is outside $(0, 0)$ also totally differentiable. In the point $(0, 0)$ we need to use the definition of the total differential, hence we need to find $Df(0, 0)$ and $\mu(h)$ in such a way that

$$f(h_1, h_2) - f(0, 0) = Df(0, 0) \cdot h + \mu(h), \qquad \lim_{h\to 0} \mu(h) = 0,$$

where $h = (h_1, h_2)$ is a vector with sufficiently small magnitude. We take $Df(0, 0) = (0, 0)$ and $\mu(h) = f(h)$, then

$$f(h_1, h_2) - f(0, 0) = f(h_1, h_2), \qquad Df(0, 0) \cdot h + \mu(h) = \mu(h),$$

and it holds

$$|\lim_{h\to 0} \mu(h)| = \lim_{h\to 0} \left| (h_1^2 + h_2^2) \sin \frac{1}{h_1^2 + h_2^2} \right| \le \lim_{h\to 0} |h_1^2 + h_2^2| = 0,$$

where we used that the absolute value is continuous, that $|\sin| \leq 1$, and the 2 cops theorem. (It would be also sufficient to show that

$$\lim_{h\to 0} \frac{f(h_1, h_2) - f(0,0) - Df(0,0)\cdot h}{|h|} = 0,$$

in which case our estimate would just be $\sqrt{h_1^2 + h_2^2}$. Try to precise this.) In this way, we have shown that Df really exists and even $Df(0,0) = (0,0)$, hence f is totally differentiable everywhere in $\mathbb{R}^2$. The partial derivatives are not continuous because for $\partial_x f$ it holds that

$$\lim_{x\to 0}\lim_{y\to 0}\left(2x\sin\frac{1}{x^2+y^2} - \frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}\right) = \lim_{x\to 0}\left(2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}\right) = -\lim_{x\to 0}\frac{2}{x}\cos\frac{1}{x^2}$$

and the last limit does not exist. In a similar fashion that also holds for $\partial_y f$ by symmetry reasons (although there we have to switch limits to $\lim_{y\to 0}\lim_{x\to 0}$, otherwise the outcome would be zero).

Remark: In this example it really is enough to take limits one by one, meaning first $y \to 0$ and then $x \to 0$, since we want to show that something is *not* continuous. Hence, it is for us enough to choose one specific direction, which in this case is along the axes. If we would like to show that something *is* continuous, this would not be enough, see the example from the first lecture.

Moreover, being continuous and having partial derivatives is NOT the same as continuous partial derivatives, which means that the partial derivatives itself must be continuous. Indeed the function above is such an example: it is continuous (prove it or use that totally differentiable functions are continuous), and it has all partial derivatives, but those are *not* continuous.