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Mathematical analysis II

A word to gradients and derivatives

Recall that for a function $f : \mathbb{R}^2 \to \mathbb{R}$, the gradient is defined as

$$\nabla f = (\partial_x f, \partial_y f).$$

This is a vector with two components that takes as arguments two variables. The geometric interpretation is that the vector, in each point, points in the direction of the steepest ascent resp. the direction of the maximum rate of increase of f.

Imagine standing in a room. Each point in space (x, y, z) has a temperature $T(x, y, z)$. The gradient $\nabla T = (\partial_x T, \partial_y T, \partial_z T)$ points from cool regions to warm ones, and in turn, the negative $-\nabla T$ points from warmer regions to cooler ones. In physics this is known as Fourier's law: the heat flux vector $q = -\nabla T$ says that heat flows from warm regions to colder ones (that's the minus sign in there).

In terms of a profile, imagine $H(x, y)$ is a landscape that assigns to every point $(x, y) \in \mathbb{R}^2$ a specific hight H. The gradient ∇H points into the direction of highest inclination, that is, out of valleys and towards mountain peaks. The magnitude gives the steepness of the slope. If you have ever seen car signs on streets with writing "40 %", this is the steepness of the mountain and can be interpreted as the magnitude of a gradient having a certain angle.

Unfortunately, for *curves* $g : \mathbb{R} \to \mathbb{R}^2$, this interpretation somehow fails. First, recall that the derivative of a curve is

$$g'(t) = (g_1'(t), g_2'(t)).$$

It does not tell us where the steepest ascent is, but rather how the tangent line on g looks like, and also with what speed we are moving in which direction *on that curve*.

However, there is still a connection between gradients and derivatives:

Let's consider a landscape with hills, described by a function $f : \mathbb{R}^2 \to \mathbb{R}$, and take a slice of it. This slice has a boundary that is a curve (so-called level set) and can be represented as a function $g : \mathbb{R} \to \mathbb{R}^2$ (we walk "around" the hill on one specific hight). The gradient ∇f points towards the hill's peak and thus *away* from the curve; in other words, the gradient ∇f and the derivative g' are perpendicular to each other. This looks like in Figure 1.

In general, Figure 2 gives an overview on what is possible and what isn't.

With directional derivatives we will work later on. A good overview on the interpretation is the Wikipedia page on gradients: `https://en.wikipedia.org/wiki/Gradient#Motivation`.

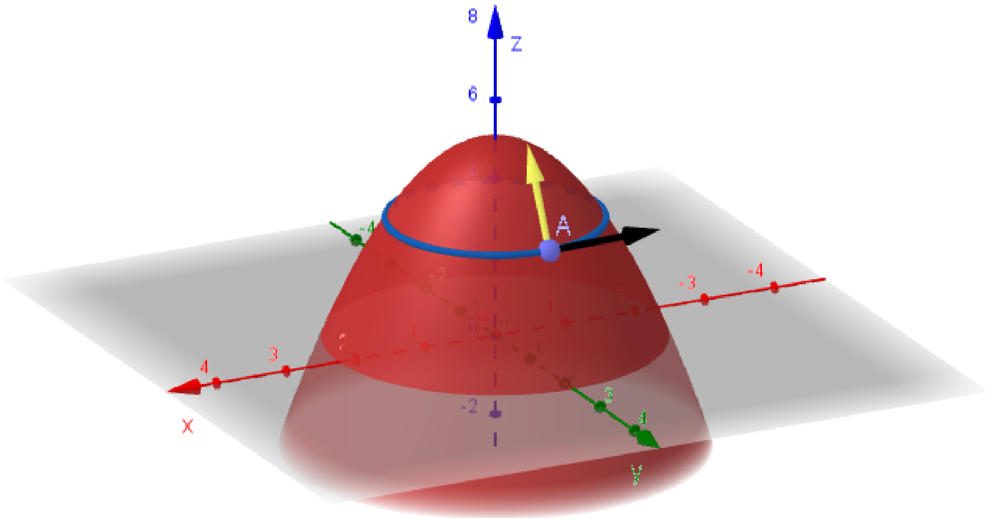

Figure 1: Red: surface, function f; blue line: level set, function g; black arrow: tangent of g on point A, i.e., $g'(t_A)$ at time t_A when g reaches A; yellow: gradient of f at point A, i.e., $\nabla f(x_A, y_A)$. Note that the yellow and black arrows are perpendicular to each other.

Summary Table

Function Type	Gradient	Directional Derivative
$f : \mathbb{R}^n \to \mathbb{R}$	Vector, points steepest ascent	Scalar, rate of change in a direction
$f : \mathbb{R} \to \mathbb{R}^2$	(Not defined)	Use derivative (tangent vector)

Figure 2: Relationship between gradient and derivative