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Mathematical analysis II

Collection of problems

Not for handing in

Exercise 1 (Metrics).

a) Let

$$d(x,y) = \left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right|.$$

Show that $(\mathbb{R}, d)$ is a metric space. Are the metrics d and $d_1(x,y) := |x-y|$ equivalent? Show or disprove.

b) Let (X,d) be a metric space. Find all $k \in \mathbb{R}$ such that

$$d_1(x,y) := (k-1)(k-3)d(x,y)$$

is a metric on X.

c) Let (X,d) be a metric space and define like in HW 1

$$\delta(x,y) = \min\{d(x,y), 1\}.$$

Are, in general, the metrics d and δ equivalent? Show or disprove.

Exercise 2 (Open and closed sets).

a) Give an example of a set $A \subset \mathbb{R}$ (with the euclidean metric) that is neither closed nor open. Can you also find such a set $A \subset \mathbb{R}^2$ or even $A \subset \mathbb{R}^n$?

b) Give an example of metric spaces (X,d) and (Y,e), a function $f : X \to Y$, and a closed subset $B \subset Y$ such that $f^{-1}[B]$ is not closed. Give also an example where you switch every "closed" with "open".

c) Find a function $f : \mathbb{R} \to \mathbb{R}$, a closed set $A \subset \mathbb{R}$ and an open set $B \subset \mathbb{R}$ such that $f[A]$ and $f[B]$ are neither open nor closed.

d) A space X is called *connected*, if the only sets that are both closed an open (clopen) are just $\emptyset$ and X.

   1) Give an example $X \subset \mathbb{R}$ that is not connected. Determine all clopen subsets of X.
   2) Let (X,d) and (Y,e) be metric spaces and

   $$f : X \to Y$$

   be continuous and onto (surjective). Show that if X is connected, then Y is as well.

Exercise 3 (Continuity).

a) Determine whether or not the following functions $f_i : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ are continuous. Can we extend them in such a way that the functions are continuous on the whole of $\mathbb{R}^2$?

1) $f_1(x,y) = \frac{x+y}{\sqrt{x^2+y^2}}$
2) $f_2(x,y) = \frac{x^2y^2}{x^2+y^2}$
3) $f_3(x,y) = \frac{x^2+y^2}{|x|+|y|}$

b) Let (X,d), (Y,e) be metric spaces, where $X = A \cup B$ is the union of open or closed subsets of X. Let moreover $f_A : A \to Y$ and $f_B : B \to Y$ be continuous with $f_A = f_B$ on $A \cap B$. Show that the function

$$f : X \to Y, \qquad f(x) = \begin{cases} f_A(x) & \text{if } x \in A, \\ f_B(x) & \text{if } x \in B \end{cases}$$

is continuous.

Exercise 4 (Differentiability, extrema, tangential planes).

a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be an affine mapping, that is, there is some linear mapping $L : \mathbb{R}^n \to \mathbb{R}$ such that $f(y) - f(x) = L(y-x)$. Show that f is totally differentiable on $\mathbb{R}^n$. How the total differential looks like?

b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable, $x_0 \in \mathbb{R}^n$ and $c = f(x_0)$. Show that the gradient $\nabla f(x_0)$ is perpendicular to the level set

$$N_f(c) = \{x \in \mathbb{R}^n : f(x) = c\},$$

i.e., the following holds: If $\varepsilon > 0$ and $\phi : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ is a differentiable curve with $\phi(0) = x_0$ and $\phi[(-\varepsilon, \varepsilon)] \subset N_f(c)$, then

$$\langle \phi'(0), \nabla f(x_0) \rangle = 0.$$

(Hint: Consider the function $g := f \circ \phi$.)

c) Calculate the partial derivatives up to order 2 of f_1 and f_2 from Exercise 3.

d) Determine position and kind of all extrema to the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x,y) = (x^2-1)^2 + y^4.$$

Additionally, calculate the tangent plane at the point $x_0 = (2,3)$. What happens to the extrema if we consider $g(x,y) = (x^2-1)^2 + y^3$ instead?

e) Let

$$f(t) = (1+t, t^2, 1-t), \qquad g(x,y,z) = 1 + x + xyz.$$

Calculate once with and once without the help of the chain rule $D(g \circ f)(0)$.

f) Let n points $(x_1, y_1), \ldots, (x_n, y_n)$ be given. Find the equation of the line $y = ax + b$, for which the sum

$$f(a,b) = \sum_{i=1}^{n} (y_i - (ax_i + b))^2$$

becomes minimal.

Exercise 5 (Mean value theorem, second order derivatives, chain rule).

a) Show that the mean value theorem fails for functions $f : \mathbb{R}^n \to \mathbb{R}^m$ with $m \geq 2$. More precisely, given $a = 0$, $h = 2\pi$, and

$$f : \mathbb{R} \to \mathbb{R}^2, \qquad f(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix},$$

show that there does not exist a $\theta \in (0,1)$ such that $f(a+h) - f(a) = f'(a+\theta h)h$.

b) Show that for the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{else,} \end{cases}$$

we have $\partial_x \partial_y f(0,0) \neq \partial_y \partial_x f(0,0)$. Why this is not a contradiction to Schwarz' theorem?

c) A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *homogeneous of degree* $k \in \mathbb{Z}$ if for any $s \in \mathbb{R} \setminus \{0\}$, we have $f(sx_1, sx_2, \ldots, sx_n) = s^k f(x_1, x_2, \ldots, x_n)$.

   1) Give examples of functions $f : \mathbb{R}^2 \to \mathbb{R}$ that are homogeneous of degree -1, 0, 1, 2, respectively.
   2) Show: If $f : \mathbb{R}^n \to \mathbb{R}$ is partially differentiable and homogeneous of degree k, then

$$\sum_{i=1}^{n} x_i \partial_{x_i} f(x_1, \ldots, x_n) = k f(x_1, \ldots, x_n).$$

Exercise 6 (Compactness, completeness).

a) Let (X,d) be a metric space. Show or disprove: if X is compact, then it is complete.

b) Let (X,d) be a metric space. Show or disprove: if X is complete, then it is compact.

c) Let $f : \mathbb{R} \to \mathbb{R}$ be a function, $K \subset \mathbb{R}$ and $V \subset \mathbb{R}$. Find examples for the following situations:

   1) K is compact, but $f[K]$ is not.
   2) K is compact, but $f^{-1}[K]$ is not.
   3) V is complete, but $f[V]$ is not.
   4) V is complete, but $f^{-1}[V]$ is not.

d) Show or disprove: if (X,d) and (Y,e) are metric spaces, $f : X \to Y$ is continuous and X is complete, then $f[X]$ is complete.

e) Let (X,d) be a metric space and $A \subset X$. For $x \in X$ we define the distance from x to A by $\operatorname{dist}(x,A) = \inf\{d(x,y) : y \in A\}$.

   1) Show that $\operatorname{dist}(\cdot, A) : X \to \mathbb{R}$ is continuous.
   2) Let $A \subset X$ be compact and $x \notin A$. Show that $\operatorname{dist}(x,A) > 0$. Does this also hold if we replace "compact" by "closed"?

f) (a bit harder task, don't waste too much time with that one) Let $A \subset \mathbb{R}^n$, I be any (countable or uncountable) index set and $U_i \subset \mathbb{R}^n$ be open for any $i \in I$. We call $(U_i)_{i \in I}$ an *(open) covering* of A if $A \subset \bigcup_{i \in I} U_i$. Show that A is compact if and only if for any open covering $(U_i)_{i \in I}$ there is a finite subcovering, i.e., there are finitely many $U_{i_1}, \ldots, U_{i_n}$ with $i_1, \ldots, i_n \in I$ such that $A \subset \bigcup_{j=1}^n U_{i_j}$. (In this sense, "compactness" is a generalization of "finiteness").

g) Give for $(0,1)$ and $[0,\infty)$ open coverings that do not posses a finite subcovering.

Exercise 7 (A mixed problem).
Let $X = (0,1)$ and

$$d(x,y) = \begin{cases} |x-y| + \frac{1}{x} + \frac{1}{y} + \frac{1}{1-x} + \frac{1}{1-y} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

a) Show that d is a metric on X.

b) Show that the space $(X, |x-y|)$ is not complete, but (X,d) is complete.

c) Are the metrics $|x-y|$ and $d(x,y)$ equivalent?

d) Is $(0,1)$ bounded/closed/open/compact in the metric d?

Exercise 8 (Taylor polynomial, implicit function theorem, inverse function theorem).

a) Find the Taylor polynomials of degree two and three of the function

$$f(x,y) = \frac{x-y}{x+y}$$

in the point $(1,1)$.

b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,p) = x^2 + px - 1$. Aim of this exercise is to train implicit function theorem (IFT) with a known example.

   1) Show directly (completing the square) that the equation is solvable for x, i.e., there is a function g_1 such that $f(x,p) = 0 \Leftrightarrow x = g_1(p)$.
   2) Use IFT to show that there exists a function g_2 such that $f(x,p) = 0 \Leftrightarrow x = g_2(p)$ for all (x,p) in some neighborhood of $(1,0)$.
   3) Differentiate the equation $f(g_2(p),p) = 0$ wrt. p using chain rule to obtain an equation for $g_2'(p)$.
   4) Check whether also g_1 fulfils this equation.

c) Show that the system of equations

$$\begin{aligned} x^2 + y^2 &= 2uv, \\ x^3 + y^3 &= v^3 - u^3 \end{aligned}$$

defines in some neighborhood of the point $(x_0,y_0) = (-1,1)$ implicitly a function $g(x,y) = (u(x,y), v(x,y))$ with $g(-1,1) = (1,1)$. Determine the Jacobi matrix of g in $(-1,1)$.

d) (A student's logarithmic rule) Show that there is a continuously differentiable function g defined in a neighborhood of $x = 1/e$ such that

$$\log(g(x) - x) = \frac{\log g(x)}{\log x}.$$

Determine first a "good" value y_0 such that $g(1/e) = y_0$. (For "generalizers": Take $1/e^r$ with $r \geq 1$ instead of $1/e$.)

e) Find for the function

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \qquad f(x, y) = (e^x \sin y, e^x \cos y)$$

an open set $U \subset \mathbb{R}^2$ such that the restriction $f|U$ is injective. Is f globally invertible?

f) Let $f(x, y, z) = (x + y + z, xy + xz + yz, xyz)$. Show that f is continuously differentiable and determine the Jacobi matrix. Decide also whether or not the inverse function theorem is applicable in the points $(1, 1, 0)$ and $(1, -1, 0)$.

g) Let

$$f(x) = \begin{cases} x + 2x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is differentiable and that the derivative is bounded on the interval $(-1, 1)$. Show moreover that in no neighborhood of $x = 0$, the function is injective. Why this isn't a contradiction with the inverse function theorem?

Exercise 9 (Directional derivatives, extrema under constraints).

a) Calculate the directional derivative of $f(x, y, z) = e^{xyz}$ in the point $x_0 = (1, 1, 1)$ with direction $v = (1, 2, -1)$. Show that indeed $D_v f(x_0) = \nabla f(x_0) \cdot v$.

b) Let

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is discontinuous in $(x, y) = (0, 0)$ (in particular, it is not totally differentiable there), but all directional derivatives exist there.

c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be totally differentiable in a point $x_0 \in \mathbb{R}^2$, and let this total differential be nonzero. Let $v, w \in \mathbb{R}^2$ such that $D_v f(x_0) = D_w f(x_0) = 0$. Show that v and w are linearly dependent. Does this also hold when replacing $\mathbb{R}^2$ with $\mathbb{R}^3$ (or even $\mathbb{R}^n$)?

d) Define

$$M = \{(x, y) \in \mathbb{R}^2 : x = y \text{ and } x \neq 0\},$$

and let

$$f(x, y) = \begin{cases} e^x - 1 & \text{if } (x, y) \in M, \\ 0 & \text{else.} \end{cases}$$

1) Show that f is partially differentiable in $(x, y) \in \mathbb{R}^2$ if and only if $(x, y) \notin M$.
2) The directional derivative $D_v f(0)$ exists for any $v \in \mathbb{R}^2$.
3) There is some $v \in \mathbb{R}^2$ with $|v| = 1$ such that $D_v f(0) \neq \nabla f(0) \cdot v$.

e) Let $f(x, y) = 4x^2 - 3xy$. Find all extrema of f in the disc $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Argue why such extrema do exist. (How to: first find the extrema inside D in the usual way, then find extrema on the boundary using constraints.)

Exercise 10 (Uniform continuity, Riemann integral 1D).

a) Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous. Show that there is some $M \geq 0$ such that for any $x \in \mathbb{R}$, we have $|f(x)| \leq M(1 + |x|)$.

b) Construct an example of a function $f : [0, 1) \to \mathbb{R}$ such that f is continuous but not uniformly continuous.

c) Prove: Let (a, b) be a bounded open interval. A continuous function $f : (a, b) \to \mathbb{R}$ is uniformly continuous if and only if we can extend it to a function that is continuous on the closed interval $[a, b]$.

d) Show that for any Riemann integrable function $f : [a, b] \to \mathbb{R}$, it holds that

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

e) Let $x_0 \in (0, 1)$ and define

$$\chi_{x_0}(x) = \begin{cases} 1 & \text{if } x = x_0, \\ 0 & \text{else.} \end{cases}$$

Show via definition that $\int_0^1 \chi_{x_0}(x)\, dx = 0$.